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Nonlinear instability of viscous plane Couette flow Part 1. Analytical approach to a necessary condition

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We perform a two-dimensional analytical stability analysis of a viscous, unbounded plane Couette flow perturbed by a finite-amplitude defect and generalize the results obtained in the inviscid limit by Lerner and Knobloch. The dispersion relation is derived and is used to establish the condition of marginal stability, as well as the growth rates at different Reynolds numbers. We confirm that instability occurs at wavenumbers of the order of ϵ, the non-dimensional amplitude of the defect. For large enough ϵR (R being the Reynolds number based on the width of the defect), the maximum growth rate is about $\frac{1}{2}\epsilon$, at approximately half the critical wavenumber. We formulate the instability conditions in the case where the flow has a finite extension in the downstream direction. Instability appears when ϵ is greater than $R_L^{-\frac{1}{3}}$, where R_L is the Reynolds number based on the downstream scale, and when the ratio of the defect width to the downstream scale lies in the interval $[(\epsilon R_L)^{-\frac{1}{2}}, \epsilon]$.

1. Introduction

Shear flows are of major interest in many astrophysical and geophysical situations because of the various instabilities they are likely to undergo, which may lead to turbulence and thus to enhanced transport. Despite their apparent simplicity, these flows often exhibit complex behaviour which is not fully understood; many of them are known to be unstable with respect to finite-amplitude perturbations under conditions where the linear theory predicts stability.

Various techniques have been applied, with more or less success, to investigate such finite-amplitude instabilities, but they have failed so far with the plane Couette flow: a plane parallel stream of constant shear (i.e. of constant vorticity). One reason is that this flow has no linear instability from which one could start to explore the nonlinear regime, as was done for instance by Zahn *et al.* (1974) when dealing with plane Poiseuille flow. A step forward was made recently by Lerner & Knobloch (1988, hereinafter referred to as LK) who studied the influence of a small defect in an inviscid Couette flow on the onset of instability. The defect is chosen in such a way that the profile of the flow, originally purely linear, presents a local maximum of vorticity and thus satisfies the Rayleigh–Fjørtøft necessary condition for instability. It is then possible to perform a *linear* analysis of the stability of the resulting flow, given the scaled amplitude ϵ of the defect. The main result of this analysis is the existence of a long-wave instability, with the critical (downstream) wavenumber k_c of order ϵ; the corresponding solutions have a growth rate of order ϵ.

More recently, Nagata (1990) produced for the first time some evidence of the existence of three-dimensional finite-amplitude solutions in plane Couette flow. These solutions are expressed as truncated modal expansions; they are obtained numerically by extending the bifurcation problem of a circular Couette system between corotating cylinders with a narrow gap to the case with zero average rotation. The critical Reynolds number for the appearance of those finite-amplitude solutions is found to be of the order of 1000.

In contrast to Nagata, we are interested in the behaviour at large Reynolds numbers, which are more relevant to geophysical and astrophysical flows. For the same reason, we shall deal with an unbounded Couette flow. The purpose of this paper is therefore to extend LKs work in the two-dimensional case by including viscous dissipation, in order to estimate the instability threshold in terms of the Reynolds number characterizing the flow. Obviously, viscosity will have a stabilizing effect by smoothing the original defect, therefore suppressing the very source of the instability. For instability to occur, one must thus require the growth rate of the instability ($O(\epsilon)$) to be greater than the viscous damping rate associated with the defect, which is ($O(1/R)$) (R being the Reynolds number of the flow, based on the size of the defect). In other words, the amplitude of the perturbation, measured as the relative increase of vorticity, must be at least of the order of $1/R$ to allow the instability.

But this condition may not be sufficient, since viscosity will also operate on the two-dimensional perturbation imposed on the mean flow. It remains to be checked whether the viscous decay of the perturbation will not be faster than that of the defect, as would be the case if the perturbation develops scales that are shorter than the width of the defect. To rule out this possibility, we shall solve the linear stability problem, including viscous dissipation. We shall assume that the decay rate of the defect can be neglected compared with the growth rate of the linear mode, and we shall check afterwards the validity of that assumption.

2. Derivation of the dispersion relation

2.1. *The model*

We examine the stability of a two-dimensional, incompressible, plane-parallel shear flow, with velocity of the form $\boldsymbol{U} = (U(Y;\epsilon), 0)$, where ϵ indicates the non-dimensional amplitude of the defect and (X, Y) are the coordinates parallel and transverse to the shear. The flow is unbounded in the Y-direction, both for simplicity and to avoid any effect of the boundaries on the instability. The model we are going to use is a discrete profile which is continuous but which contains discontinuities in its first derivative (Model B of LK). This choice is motivated by the fact that this problem can be solved by elementary methods (Drazin & Reid 1981). Furthermore, LKs careful analysis showed that results obtained with this kind of profile remain qualitatively and even quantitatively true for similar but smooth, continuous profiles.

The profile of the flow is thus taken to be, in non-dimensional variables:

$$u(y;\epsilon) = \begin{cases} y-\epsilon & \text{if } y \leqslant -1 \text{ (region I)}; \\ (1+\epsilon)\,y & -1 \leqslant y \leqslant 1 \text{ (region II)}; \\ y+\epsilon & \text{if } y \geqslant 1 \text{ (region III)}. \end{cases} \tag{1}$$

The unit of length is the width d of the defect, and the unit of time is the inverse of the shearing rate $\mathrm{d}U/\mathrm{d}Y$ in the unperturbed flow (regions I and III).

2.2. *The perturbation equation*

As usual when dealing with plane-parallel flows, we factorize the stream function of the perturbation as

$$\psi = \mathrm{Re}\,\{\phi(y)\exp[\mathrm{i}k(x-ct)]\};$$

the function ϕ then satisfies the Orr–Sommerfeld equation (see Drazin & Reid 1981):

$$\mathrm{i}k[(u-c)\,\mathscr{L}\phi-u''\phi]=\frac{1}{R}\mathscr{L}^2\phi,\quad \text{where}\quad \mathscr{L}=\frac{\partial^2}{\partial y^2}-k^2; \tag{2}$$

R is the Reynolds number of the perturbed flow based on the width d of the defect:

$$R=\frac{\mathrm{d}U}{\mathrm{d}Y}\frac{d^2}{\nu},$$

k is the wavenumber of the perturbation in the flow direction and c is its phase velocity. This equation, completed with the boundary conditions for an unbounded flow

$$\phi(\pm\infty)=0,\quad \frac{\partial}{\partial y}\phi(\pm\infty)=0,$$

constitutes an eigenvalue problem for the (complex) phase speed $c(k,\epsilon)$. Its solution can be found by solving (2) in each of the regions I, II and III, and then applying some jump conditions on ϕ across $y=\pm1$ to establish the dispersion relation for c. Drazin (1961) showed that these jump conditions are

$$\left.\begin{aligned}
&\frac{1}{kR}\Delta\phi=0,\\
&\frac{1}{kR}\Delta\left[\frac{\partial\phi}{\partial y}\right]=0,\\
&\frac{1}{kR}\Delta\left[\frac{\partial^2\phi}{\partial y^2}+\mathrm{i}kR(u-c)\,\phi\right]=0,\\
&\frac{1}{kR}\Delta\left[\frac{\partial^3\phi}{\partial y^3}-\mathrm{i}kR\left\{(u-c)\frac{\partial\phi}{\partial y}-u'\phi\right\}\right]=0.
\end{aligned}\right\} \tag{3}$$

These conditions are valid for any kR: in the limit $kR\rightarrow\infty$, one retrieves the classical inviscid jump conditions.

2.3. *Simplification of the problem*

If one proceeds as just stated, one has to calculate an 8×8 determinant, which is a formidable task even in the limit of small k and $1/R$. Fortunately, it is possible to simplify the problem by looking only for symmetric solutions.

The flow profile (1) being antisymmetric, we observe that (2) is invariant under the transformation $y\rightarrow-y$, $c\rightarrow-c^*$ and $\phi\rightarrow\phi^*$ (Tatsumi & Gotoh 1960). It has been established by Pekeris (1936) that for such a profile the most unstable (or the least stable) solution is unique, and therefore has a zero phase velocity ($c=-c^*$) and is symmetric ($\phi(y) = \phi^*(-y)$). This property holds to leading order in $(kR)^{-1}$. For higher kR, the symmetry is broken, and solutions appear that have non-zero phase velocities. Since we are seeking the most unstable modes, we shall assume from now on that the solution is indeed symmetric, which reduces the problem to the computation of a 4×4 determinant.

2.4. *Solving the problem*

The second derivative of u being zero, the Orr–Sommerfeld equation can be put in a very simple form. If in each domain we let

$$\eta(y) = \frac{u(y)-c}{u'(y)}, \quad \beta = (\mathrm{i}kRu')^{-\frac{1}{3}} \quad \text{with} \quad \mathrm{ph}\,\beta = -\tfrac{1}{6}\pi,$$

(2) becomes

$$[\beta^3(\mathrm{D}^2-k^2)-\eta](\mathrm{D}^2-k^2)\phi = 0,$$

where D stands for $\mathrm{d}/\mathrm{d}\eta$. Note that the variable η is now also discontinuous. We have the following limits:

$$\left.\begin{aligned} &\lim_{y\to-1^-}\eta(y) = -1+\epsilon-c \quad \text{and} \quad \lim_{y\to-1^+}\eta(y) = \frac{-1+\epsilon-c}{1+\epsilon}, \\ &\lim_{y\to1^+}\eta(y) = 1+\epsilon-c \quad \text{and} \quad \lim_{y\to1^-}\eta(y) = \frac{1+\epsilon-c}{1+\epsilon}. \end{aligned}\right\} \tag{4}$$

The boundary conditions are now

$$\phi(\eta) = \mathrm{D}\phi(\eta) = 0 \quad \text{when} \quad |\eta|\to\infty. \tag{5}$$

We solve (3) following the classical method (Mises 1921*a*, *b*; Hopf 1914). The solutions of $[\beta^3(\mathrm{D}^2-k^2)-\eta]F = 0$ are the Airy functions $A_j(z)\,(j = 1,2,3)$ of argument $z = (\zeta+k^2\beta^2)$, where $\zeta = \eta/\beta$ (their properties are summarized in the Appendix). Therefore the solutions of (4) are linear combinations of $\exp(\pm ky)$ and of two particular solutions Ψ_j of the inhomogeneous equation $(\mathrm{D}^2-k^2)\psi_j = \beta^{-4}A_j(\zeta+k^2\beta^2)$, which can be found by the method of variation of parameters:

$$\psi_j(\eta) = \frac{1}{k\beta^4}\int_{\infty_j}^{\eta} \sinh k(\eta-\eta')\,A_j(\zeta'+k^2\beta^2)\,\mathrm{d}\eta', \tag{6}$$

with $\infty_j\,(j = 1,2,3)$ being the path of integration tending to infinity in the sectors S_j defined in the Appendix. Note that the scale of variation of ψ_j is that of the argument of the Airy function, that is mainly $\beta = (kR)^{-\frac{1}{3}}$.

It remains to choose the solutions $\Phi(\eta)$ which satisfy $\phi(\eta(-y)) = \phi^*(\eta(y))$ and the boundary conditions (5) at infinity:

$$\phi(\eta) = \begin{cases} \alpha\exp \mathrm{k}y + \delta\Psi_2(\eta) & \text{for} \quad y\leqslant -1; \\ b\cosh \mathrm{k}y + \mathrm{i}b\tan(\chi)\sinh ky + \gamma\Psi_3(\eta) & \text{for} \quad -1\leqslant y\leqslant 1; \\ \alpha^*\exp -ky + \delta^*\Psi_1(\eta) & \text{for} \quad y\geqslant 1; \end{cases} \tag{7}$$

where α, δ and γ are complex numbers, while b and χ are real.

Because of the symmetry properties of these functions, the jump conditions at $y = -1$ and at $y = 1$ are complex conjugates of one another. We can then consider χ

as an arbitrary phase and impose the jump conditions only at $y = -1$ (say). This procedure gives a relation between c, k and χ. The dispersion relation is then obtained by eliminating χ between this relation and its complex conjugate and can be written:

$$4k^2|D_1|^2 + (kR\epsilon)^2(1-\mathrm{e}^{-4k})\,|kD_3 - D_2|^2 - 4k^2\epsilon R\,\mathrm{Im}\,[(kD_3 - D_2)\,D_1^*] = 0, \tag{8}$$

where the D_j are the following 2×2 determinants:

$$\left.\begin{aligned} D_1 &= \begin{vmatrix} \beta^{-4}A_2(\zeta_-) & -\tilde{\beta}^{-4}A_3(\tilde{\zeta}_-) \\ \beta^{-5}A_2'(\zeta_-) & -\tilde{\beta}^{-5}A_3'(\tilde{\zeta}_-) - kR\epsilon\Psi_3(\tilde{\zeta}_-) \end{vmatrix}, \\ D_2 &= \begin{vmatrix} \Psi_2'(\zeta_-) & -\Psi_3'(\tilde{\zeta}_-) \\ \beta^{-4}A_2(\zeta_-) & -\tilde{\beta}^{-4}A_3(\tilde{\zeta}_-) \end{vmatrix}, \\ D_3 &= \begin{vmatrix} \Psi_2(\zeta_-) & -\Psi_3(\tilde{\zeta}_-) \\ \beta^{-4}A_2(\zeta_-) & -\tilde{\beta}^{-4}A_3(\tilde{\zeta}_-) \end{vmatrix}, \end{aligned}\right\} \tag{9}$$

where $\zeta_- = -(1+\epsilon+\mathrm{i}c_i)/\beta$, $\tilde{\zeta}_- = \zeta_-/(1+\epsilon)^{\frac{2}{3}}$ and $\tilde{\beta} = \beta(1+\epsilon)^{\frac{1}{3}}$. It can be checked that in the limit $kR \to \infty$, this dispersion relation reduces to the inviscid one obtained by LK.

The dispersion equation (8) depends on four parameters, namely k, R, ϵ and c_i. It is therefore almost impossible to explore the whole range of parameters, and so we shall confine our domain of exploration to wavenumbers of order ϵ, where LK located the inviscid instability. It is then convenient to express all quantities in terms of the 'natural variables' $\tilde{k} = k/\epsilon$ and $\tilde{R} = \epsilon R$ which are both of order unity. In the limit of small ϵ, all the functions involved in the determinants can be expanded in power of ϵ. Keeping only leading orders in ϵ and turning back to our original variables, we thus obtain the following dispersion relation:

$$\frac{k}{\epsilon} = \mathrm{Re}\left(\frac{1}{\beta}B_1(\zeta_-)\right), \tag{10}$$

where $\zeta_- = -(1+\epsilon+\mathrm{i}c_i)/\beta$ and $\beta = (kR)^{-\frac{1}{3}}\mathrm{e}^{-\mathrm{i}\pi/6}$and B_1 is a complex function whose definition and properties are given in the Appendix. The dispersion relation given by (10) can be studied both analytically and numerically. The results are presented in the following section.

3. Results

3.1. *Marginal stability*

The curve of marginal stability $k_c(\omega = 0) = f[\epsilon R]$ has been computed numerically. The result is given in figure 1. In both limits $\epsilon R \to 0$ and $\epsilon R \to \infty$, its analytical form can be found using the properties of the function B_1 which are provided in the Appendix. This gives

$$\left.\begin{aligned} k_c &\approx \epsilon && \text{when} \quad \epsilon R \to \infty; \\ k_c &\approx \epsilon[B_1(0)\sin(\tfrac{1}{3}\pi)]^{\frac{3}{2}}(\epsilon R)^{\frac{1}{2}} && \text{when} \quad \epsilon R \to 0, \end{aligned}\right\} \tag{11}$$

with $B_1(0) \approx 1.1$. The value reached in the $\epsilon R \to \infty$ limit is hardly surprising: it is the result obtained by LK in the inviscid case.

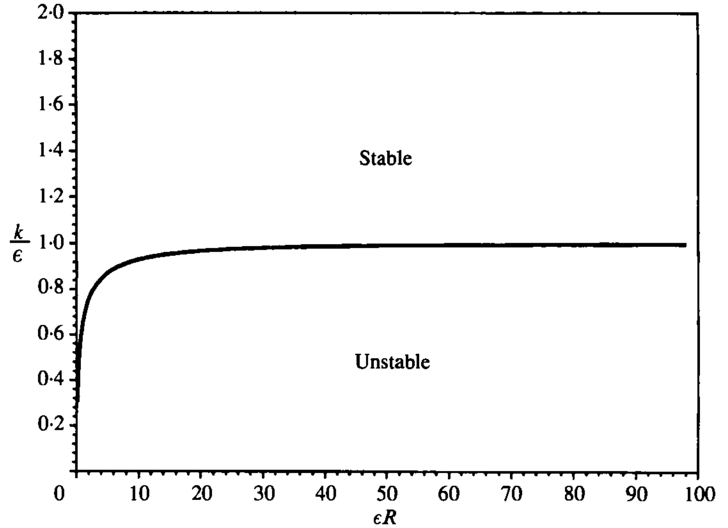

FIGURE 1. Curve of marginal stability.

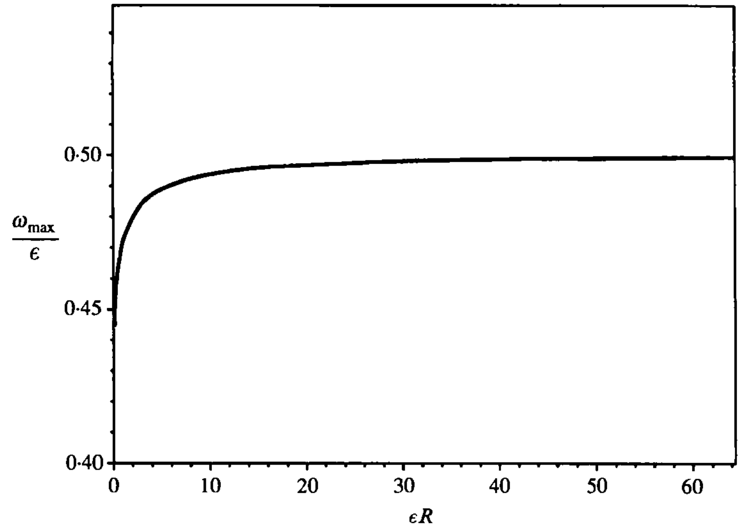

FIGURE 2. Maximal growth rate ω/ϵ versus Reynolds number ϵR.

3.2. *Maximum growth rate*

The maximum growth rate ω_{max}/ϵ has been computed for several values of the rescaled Reynolds number ϵR. The result is displayed in figure 2. Let us recall that our analysis is valid only for $\epsilon R \gg 1$, which guarantees that the defect will last long enough for the perturbation to develop. In the inviscid limit $\epsilon R \rightarrow \infty$, the maximum growth rate tends to $\frac{1}{2}$, as expected from the results of LK. It is quite insensitive to the strength of the viscosity: even for values of ϵR as small as 0.1, ω_{max}/ϵ is still 0.44. However, the value $k = k_{max}$ at which this maximum growth rate occurs decreases from 0.5ϵ (inviscid limit) to 0 (infinite viscosity limit). The critical wavenumber k_c exhibits a similar decrease.

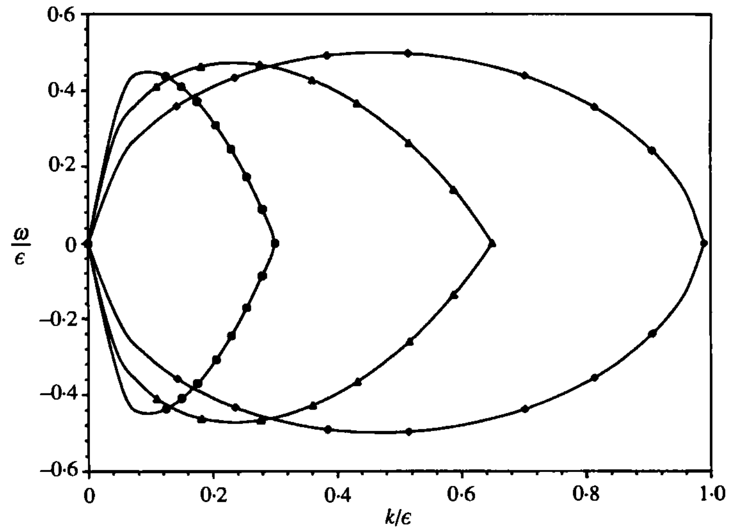

FIGURE 3. Growth rate ω/ϵ versus wavenumber k/ϵ at different Reynolds numbers: $\epsilon R = 0.01$ (circles); $\epsilon R = 1$ (triangles); $\epsilon R = 50$ (squares).

ϵR	ω_{max}/ϵ	k_{max}/ϵ	$k_c/2\epsilon$
0.1000000	0.4444663	7.9999998×10^{-2}	0.1514543
0.5680000	0.4654000	0.2000000	0.2787105
1.000000	0.4722691	0.2400000	0.3247618
4.960000	0.4888077	0.3600000	0.4352087
8.920000	0.4929987	0.4000000	0.4612517
50.50000	0.4988920	4.4800000	0.4950908

TABLE 1. Maximum growth rate, wavenumber at which it occurs, and critical wavenumber, versus Reynolds number

Some numerical results are given in table 1. For reference, we also give in figure 3 the curve $\omega/\epsilon = f(k/\epsilon)$ for different ϵR. Those curves illustrate the influence of viscosity on the growth rate, the case $\epsilon R = 50$ (squares) corresponding to almost the inviscid limit.

4. Discussion

Our main result is that the maximum growth rate of the perturbation is little affected by viscous damping. Therefore, we confirm the instability condition suggested in the introduction: that the growth rate of the perturbation be larger than the decay rate of its cause (the finite-amplitude defect).

So far, we have considered a Couette flow that is unbounded in both directions. We insist on this property in the cross-stream direction, because we want to avoid boundaries which might play an active role in the instability, in order to focus only on the effect of the profile defect. Note that this assumption is consistent with the results, since the perturbation decays exponentially far from the defect.

But the situation is quite different in the direction of the flow, in which our solutions are assumed to have a periodic behaviour. In most cases of interest, there is a maximum scale allowed in that direction. When performing numerical simulations, such a finite scale is imposed by the computational domain. In a rotating Couette flow, to which this analysis can be extended in the narrow gap limit, the circular topology introduces the circumference as the limit length. The existence of such a minimum wavenumber has a direct impact on the threshold of the finite-amplitude instability, as we shall see next.

We assume that the maximal downstream size is $2\pi L$ and we call d the width of the defect, as before. In the non-dimensional units introduced in §2, the minimum wavenumber is then such that $k_{\min}(2\pi L/d) = 2\pi$. To derive the instability condition, we thus identify the critical wavenumber with $k = k_{\min} = \Delta$, where $\Delta = d/L$ measures the relative defect width, and we introduce R_L, the Reynolds number characterizing the flow based on the maximum downstream scale L,

$$R_L = \frac{\mathrm{d}U}{\mathrm{d}Y}\frac{L^2}{\nu}.$$

Two conditions must be satisfied. The first is that established by LK, namely that the relative amplitude of the defect be larger than its width:

$$\epsilon > \Delta.$$

The second has just been recalled above; it expresses that the growth rate of the perturbation, of order $\epsilon\,\mathrm{d}U/\mathrm{d}Y$, is larger than the decay rate of the defect, of order of ν/d^2. This translates into: $\epsilon\Delta^2 \gtrsim R_L^{-1}$. Thus the width Δ must lie in the interval

$$(\epsilon R_L)^{-\frac{1}{2}} \lesssim \Delta < \epsilon, \tag{12}$$

and this is possible only if

$$\epsilon \gtrsim R_L^{-\frac{1}{3}}. \tag{13}$$

Refining that argument further and taking into account the decay of the amplitude of the defect, Gill (1965) reached a similar formula, but with a logarithmic correction:

$$\epsilon^{-3}\log\epsilon \gtrsim R_L.$$

This correction would increase substantially the critical Reynolds number. However, not only does the amplitude of the defect decrease, but its width increases with time, an effect which was not considered by Gill. Indeed, a localized defect evolves with time as $t^{-\frac{1}{2}}\exp(-Y^2/4\nu t)$ according to the one-dimensional Orr–Sommerfeld equation. Therefore, the amplitude of the defect varies as $t^{-\frac{1}{2}}$, and its width as $t^{\frac{1}{2}}$. The viscous dissipation thus decreases with time as t^{-1}, which is faster than the $t^{-\frac{1}{2}}$ decline of the growth rate. Consequently, it suffices that our criterion for instability (13) be fulfilled at some initial time.

The necessary condition (13), which is independent of the width of the defect, thus defines the threshold of the finite-amplitude instability for large Reynolds numbers. Although the finite-amplitude perturbation we have considered is admittedly rather specialized, we conjecture that the scaling (13) derived from it ought to be more general.

A confirmation of this conjecture comes from a reinterpretation of our nonlinear instability in terms of a 'negative viscosity' instability. The energy balance is indeed rather peculiar here: the large-scale perturbations, characterized by their small wavenumber $k \sim \epsilon$, draw their energy from the small-scale velocity field associated with the defect. Therefore, energy goes from small scale to large scale, as if through the influence of a negative viscosity. As shown recently by Dubrulle & Frisch (1991), such a phenomenon can actually occur for a wide class of two-dimensional shear flows periodic in both space and time and can be proved rigorously using a multi-scale expansion. The influence of the basic velocity field on a given large-scale perturbation is then modelled by an eddy-viscosity ν^t given by

$$\nu^t = \nu - \frac{\langle \Psi^2 \rangle}{\nu}. \tag{14}$$

Here, ν is the ordinary molecular viscosity and Ψ is the stream function of the basic flow. The angular brackets represent the space–time average over the periodicities (only space average if the basic flow is time-independent). From (14), it is obvious that the (negative viscosity) instability sets in as soon as the condition

$$\nu < \langle \Psi^2 \rangle^{\frac{1}{2}} \tag{15}$$

is satisfied. We can translate this condition to the case studied in this paper if we consider the defect as the small-scale flow, the large-scale flow being the Couette flow. In that case, the relevant stream function has a characteristic lengthscale of the order of the size of the defect and a characteristic amplitude of the order $d^2 \epsilon \, \mathrm{d}U/\mathrm{d}Y$. We see then that the instability condition (12) derived for the Couette flow on a phenomenological basis is just the translation of (15), which is exact for spatially periodic shear flows. The condition $\Delta < \epsilon$ ensures that the characteristic scale of the perturbation – which is the downstream scale since the basic flow is unbounded – is larger than the characteristic scale of the defect (recall that $\epsilon < 1$ is an implicit condition of our asymptotic analysis of §2), and so that the scale separation is fulfilled.

This confirmation of our instability condition (12) is reassuring. It remains to check whether such prediction, based on a necessary condition, is in agreement with the experiments. Surprisingly, very few results are available concerning instabilities in plane Couette flow. To our knowledge, no experiments have been performed since Reichardt's in 1956. Using a configuration of aspect ratio $H/L' = \frac{1}{5}$ (H and L' being respectively the size in the cross-stream and in the downstream direction of his apparatus), he found that turbulence occurred for Reynolds numbers (based on the channel width) greater than about $R_c = 1500$. He observed that the turbulent mean flow organized itself in a slender S-shape. Assuming that the finite size of the apparatus did not alter the dynamics of the instability (e.g. that no boundary layers were present), let us estimate what would be, according to our analysis, the perturbation amplitude required to trigger the finite-amplitude instability. Since $L/H = L'/2\pi H$, we predict that at the critical downstream Reynolds number, $R_L = (0.8)^2 R_c$, the amplitude of the perturbation should be at least $\epsilon \gtrsim (R_L)^{-\frac{1}{3}} = 0.1$, implying a width of the defect of less than $d = 0.08H$. The velocity perturbation would thus be of the order of $\epsilon d/H = 0.008$, thus about 0.8% of the basic velocity. This value seems quite plausible, if we interpret it as the level of the fluctuations generated in the experiment, which unfortunately could not be determined by Reichardt. Modern experiments on plane Couette flow would therefore be most welcome, with direct measurements of the instability threshold as well as finer descriptions of the structure of the turbulent regime.

Another way to obtain this information is to perform numerical simulations. A few years ago, Orszag & Kells (1980) showed that a possible scenario leading to turbulence in linearly stable flows was to combine a two-dimensional decaying mode of finite amplitude with an infinitesimal, three-dimensional perturbation. In their numerical simulation of plane Couette flow, they observed that the small perturbation would grow exponentially above some critical Reynolds number, but the low spatial resolution prevented them from pursuing the calculation into a more developed phase. They also noticed the important role of the inflexion points occurring in the mean flow profile.

Here, we are suggesting a different scenario, namely that a one-dimensional finite-amplitude perturbation of the mean flow should be sufficient to lead to instability. Such instability will be two-dimensional at the beginning, but it is likely to become three-dimensional after a finite time, as observed in mixing layers, both in the laboratory and in numerical simulations (Lesieur *et al.* 1988). We are currently performing high-resolution computer simulations to investigate the nonlinear evolution of such a finite-amplitude defect (Dubrulle 1991).

5. Conclusion

In this paper, we extended to viscous flows the result obtained by Lerner & Knobloch (1988) in the inviscid limit. We formulated the instability conditions in the (realistic) case where the flow has a finite extent, $2\pi L$, in the downstream direction. As expected, LK's necessary condition also holds in the viscous regime, namely that the relative maximum of vorticity characterizing the defect, ϵ, be larger than its relative width in the mean flow profile, d/L. In addition, we showed that this magnitude ϵ must also be larger than $R_L^{-\frac{1}{3}}$, R_L being the Reynolds number of the mean flow based on the downstream scale L. For a given ϵ satisfying these conditions, $\epsilon > R_L^{-\frac{1}{3}}$, the profile defect which leads to instability must have a relative width in the interval

$$(\epsilon R_L)^{-\frac{1}{2}} < \frac{d}{L} < \epsilon.$$

Such finite-amplitude instabilities of shearing flows may well be the cause of the turbulence invoked in various geophysical and astrophysical situations to account for enhanced transport. In particular, we believe that the 'turbulent viscosity' in accretion discs, which is responsible for the conversion into heat of the gravitational energy of the accreted matter, is due to such an instability arising in quasi-keplerian rotation (Zahn 1984). Such instabilities are also likely to occur in differentially rotating stars, where they will contribute to the vertical transport of chemicals and of angular momentum (Zahn 1975).

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Appendix

In this Appendix, we summarize the definitions and the properties of the Airy functions which have been used in the course of this paper. A more complete account can be found in the Appendix A of Drazin & Reid (1981), from which most of the results below have been taken.

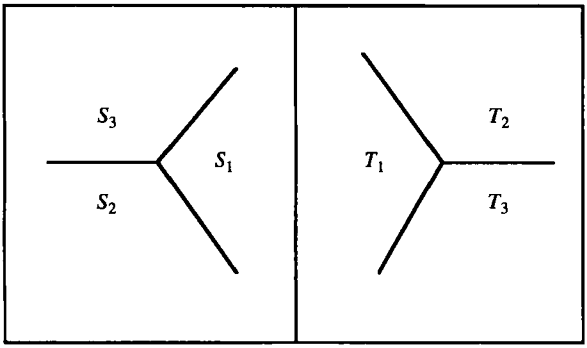

FIGURE 4. Sectors S_j and T_j used in the Appendix.

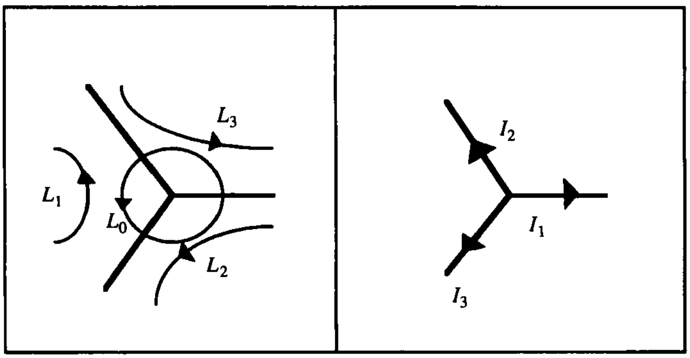

FIGURE 5. The paths of integration in the T-plane.

A.1. *The Airy functions $A_j(z)$*

The Airy functions $A_j(z)$ ($j = 1, 2, 3$) are solution of Airy's equation $f'' - zf = 0$ defined such that A_j is recessive (exponentially decaying) in the sector S_j delineated in figure 4. Any two of these functions form a pair of linearly independent solutions of Airy's equation; they are related by the connection formula

$$\sum_{j=1}^{3} A_j(z) = 0, \tag{A 1}$$

and their Wronskians are

$$W(A_1, A_2) = W(A_2, A_3) = W(A_3, A_1) = -\tfrac{1}{2}\pi\mathrm{i}. \tag{A 2}$$

More specifically, A_2 and A_3 are related to A_1 through the rotation formulae

$$A_2(z) = \mathrm{e}^{2\pi\mathrm{i}/3} A_1(z\,\mathrm{e}^{2\pi\mathrm{i}/3}), \quad A_3(z) = \mathrm{e}^{-2\pi\mathrm{i}/3} A_1(z\,\mathrm{e}^{-2\pi\mathrm{i}/3}). \tag{A 3}$$

A.2. *The functions $A_j(z;p)$*

These functions have been introduced by Reid (1974) to deal with inner and outer expansions. The functions $A_j(z;p)$ are solutions of the differential equation

$$(\mathrm{AD} + p - 1)f = 0, \tag{A 4}$$

where $\mathrm{D} = \mathrm{d}/\mathrm{d}z$ and $\mathrm{A} = \mathrm{D}^2 - z$. For $p = 0$, one has $A_j(z;0) = A_j(z)$. All we have to know here are the following properties of the functions of degree 1:

$$A_j(0;1) = -\tfrac{1}{3} \tag{A 5}$$

and
$$A_j(z,1) = \int_{\infty_j}^{z} A_j(t)\,\mathrm{d}t, \tag{A 6}$$

where ∞_j denotes a path of integration that tends to infinity in the sector S_j.

A.3. *The functions $B_j(z)$*

These functions are the solutions of the inhomogeneous Airy equation $f''-zf = 1$, and the index j refers here to the sector T_j shown in figure 4 in which B_j is well balanced. They satisfy the same rotation formulae as the $A_j(z)$, and are related to them by the three connection formulae:

$$\left.\begin{aligned} B_1(z)-B_2(z) &= 2\pi\mathrm{i}A_3(z),\\ B_2(z)-B_3(z) &= 2\pi\mathrm{i}A_1(z),\\ B_3(z)-B_1(z) &= 2\pi\mathrm{i}A_2(z).\end{aligned}\right\} \tag{A 7}$$

A.4. *Asymptotic expansions*

We adopt the convention that $\operatorname{ph} z$ lies in the range $[-\frac{4}{3}\pi, \frac{2}{3}\pi]$. In terms of the auxiliary functions

$$A_{\pm}(z;p) = \tfrac{1}{2}\pi^{-\frac{1}{2}}(\pm 1)^p z^{-(2p+1)/4}\exp\left[\pm\xi\sum_{s=0}^{\infty}(\pm 1)^s a_s(p)\,\xi^{-s}\right], \tag{A 8}$$

where $\xi = \frac{2}{3}z^{\frac{3}{2}}$ and the $a_s(p)$ are polynomials in p of degree $2s$, with $a_0(p) = 1$, the $A_j(z;p)$ have the following asymptotic expansions for $z \gg 1$:

$$\left.\begin{aligned} A_1(z;p) &\sim A_-(z;p) \qquad (z\in T_2\cup T_3),\\ A_2(z;p) &\sim \mathrm{i}A_+(z;p) \qquad (z\in T_3\cup T_1),\\ A_3(z;p) &\sim \begin{cases} -A_-(z;p) & (z\in T_1),\\ -\mathrm{i}A_+(z;p) & (z\in T_2).\end{cases}\end{aligned}\right\} \tag{A 9}$$

In the other sectors, one has to use the rotation formulae (A 3). The asymptotic expansion for the $B_j(z)$ is

$$B_j(z) \sim (-1)\,z^{-1}\{1-\tfrac{1}{3}(-1)(-2)(-3)\,z^{-3}+\ldots\} \quad (z\in T_j). \tag{A 10}$$

The asymptotic expansion of the B_j in the other sectors can also be found by the rotation formulae (A 3).

A.5. *Integral representations*

The $A_j(z;p)$ and the $B_j(z)$ admit integral representation in the form

$$\left.\begin{aligned} A_j(z;p) &= \frac{1}{2\pi\mathrm{i}}\int_{L_j} t^{-p}\exp\left(zt-\tfrac{1}{3}t^3\right)\mathrm{d}t,\\ B_j(z) &= \int_{I_j}\exp\left(zt-\tfrac{1}{3}t^3\right)\mathrm{d}t,\end{aligned}\right\} \tag{A 11}$$

where the paths L_j and I_j are shown in figure 5. These representations can be used for numerical computation.

REFERENCES

Drazin, P. G. 1961 Discontinuous velocity profiles for the Orr–Sommerfeld equation. *J. Fluid Mech.* **7**, 401.

Drazin, P. G. & Reid, W. H. 1981 *Hydrodynamic Stability*. Cambridge University Press.

Dubrulle, B. 1991 Non-linear stability of plane Couette flow. In *Conf. on Largescale Structure in Non-linear Physics, Villefranche, France, 13–18 January 1991* (ed. J.-D. Fournier, C. Fried & P.-L. Sulem). Springer.

Dubrulle, B. & Frisch, U. 1991 The eddy-viscosity of parity invariant flow. *Phys. Rev.* A **43**, 5355.

Gill, A. E. 1965 Non-linear stability of pipe flows. *J. Fluid Mech.* **21**, 503.

Hopf, L. 1914 Der Verlauf kleiner Schwingungen auf einer Strömung reibender Flüssigkeit. *Ann. Phys. Leipzig* (4) **44**, 1.

Lerner, J. & Knobloch, E. 1988 The long wave instability of a defect in a uniform parallel shear. *J. Fluid Mech.* **189**, 117 (referred to herein as LK).

Lesieur, M., Staquet, C., Le Roy, P. & Comte, P. 1988 The mixing layer and its coherence examined from the point of view of two-dimensional turbulence. *J. Fluid Mech.* **192**, 511.

Mises, R. von 1912*a* Beitrag zum Oszillationsproblem. In *Festschrift H. Weber*, pp. 252–82. Leipzig: Teubner.

Mises, R. von 1912*b* Kleine Schwingungen und Turbulenz. *Jber. Deutsch. Math.-Verein* **21**, 241.

Nagata, M. 1990 Three-dimensional finite amplitude solutions in plane Couette flow: bifurcation from infinity. *J. Fluid Mech.* **217**, 519.

Orszag, S. A. & Kells, L. C. 1980 Transition to turbulence in plane Poiseuille and plane Couette flow. *J. Fluid Mech.* **96**, 159.

Pekeris, C. L. 1936 On the stability problem in hydrodynamics. *Proc. Camb. Phil. Soc.* **32**, 55.

Reichardt, H. 1956 Uber die Geschwindigkeitsverteilung in einer geradlinigen turbulenten Couetteströmung. *Z. angew. Math. Mech.* 26.

Reid, W. H. 1974 Uniform asymptotic approximations to the solutions of the Orr–Sommerfeld equation. Part 1. Plane Couette flow. *Stud. Appl. Maths* **53**, 91.

Tatsumi, T. & Gotoh, K. 1960 The stability of free boundary layers between two uniform streams. *J. Fluid Mech.* **7**, 433.

Zahn, J.-P. 1975 Differential rotation and turbulence in stars. *Mém. Soc. R. Sci. Liège* **8**, 31.

Zahn, J.-P. 1984 Stability of rotation laws. In *Proc. 25th Liège Intl Astrophys. Colloq.* (ed. A. Noels), p. 407. Observatoire de Liège.

Zahn, J.-P., Toomre, J., Spiegel, E. A. & Gough, D. O. 1974 Non-linear instability of plane Poiseuille flow. *J. Fluid Mech.* **64**, 319.